Design Analogy between Optimal Time-Fuel and Rate-Ledge Relay Controllers

Larry R. White*
Southern Services, Inc., Birmingham, Ala.

Bruce K. Colburn†
Texas A&M University, College Station, Texas

Joseph S. Boland III‡
Auburn University, Auburn, Ala.

An empirically obtained reaction control jet relay control law with deadband is used as the basis for determining an equivalent weighted time-fuel optimal switching curve according to a least-squares criterion. The derived transformation from the empirical to the optimal law is found to be reversible and to yield a unique transformed control law. The proposed method provides a basis for determining the behavior of an easily implemented relay control law using well-known optimal control results, as well as determining the equivalent relay law corresponding to an analytically determined optimal control law. A numerical example illustrates the transformation technique and simulation results are presented to compare the two control laws.

I. Introduction

A RATE-LEDGE relay control law used in the attitude control of spacecraft is used as the starting point for studying equivalent optimal time-fuel switching curves. The rate-ledge control law (RLC) derives its name from the sloping deadzone characteristic as shown in Fig. 1.[1,2] This is a practical minimum time-fuel control law of the type used with large vehicles of the Skylab class. The horizontal deadzone for $|\phi| > \phi_R$ is used to reduce the fuel usage required by utilizing a nonzero $\dot{\phi}$ value to cause the vehicle to drift towards the origin. The ledge with deadband serves two purposes: 1) it avoids thruster on-off chattering by forcing a finite on-finite off time thruster firing sequence; and 2) it allows for linear mixing of position and rate attitude signals.

The motivation for this study was the desire to have available a mathematically tractable technique for determining a series of simple switching curves, each representing a particular type of standard control law. The rate-ledge characteristic was obtained empirically on hundreds of computer simulations. From these studies some "pseudo-optimum" rate-ledge relay switching curves were developed for use in a conventional three-axis six-engine attitude control system. One of the obvious difficulties with empirically-based results is that often the relationship between input-output parameters is lost. Optimal control laws, on the other hand, possess well-defined input-out-ut parameter characteristics. For this reason, a general technique for obtaining a switching chracteristic based on an optimal weighted time-fuel cost function is derived by minimizing a least-squares error criterion. The reverse transformation, driving an equivalent RLC controller from an optimal weighted time-fuel law, is then developed.

Numerical values for the various parameters noted in Fig. 1 are generally obtained experimentally for a specific application. This information is then generalized from one curve and one vehicle to a class of vehicles utilizing different control weighting factors by determining an equivalent (in a mean-square sense) optimal control strategy which is well defined and for which there is a body of standardized control results available.[1,3-5] This type of study of optimal switching curves for RCJ systems continues as evidenced by Ref. 6.

It can be shown that the characteristic curve of the type in Fig. 1 may be obtained from the control block diagram shown in Fig. 2. Simplicity is the outstanding point of such a control strategy. However, without having a straightforward technique for determining the effect on response of variations in the parameters, the control law in Fig. 2 becomes of little value.

The main results of this paper are the derivation of general transformations for obtaining a unique equivalent (in the sense of time-fuel weighting) optimal switching curve for gas thruster control systems and the derivation of a reverse transformation. Existence and uniqueness of the transformations are proved, and a design procedure presented for analytically determining numerical values of all parameters. The results are available in a general form and can be investigated in the light of well-known optimal control results. Using the reverse transformation, an equivalent RLC controller for a desired optimal law may be determined which is easily implemented with off-the-shelf hardware without the need of a computer to synthesize special switching curves.

II. Problem Statement

It will be assumed that the three orthogonal axes of a space vehicle to be controlled are aligned with the vehicle Euler axes so that all angular position and velocity feedback signals are available directly from sensors. In this way no nonlinear coordinate rotations are required. The remainder of this paper will consider the control of one of these axes. The analysis is further simplified by assuming the system dynamics can be represented as s^{-2} (i.e., no aerodynamic torques). The system to be controlled is described by the following state equations

$$\dot{x}_1 = x_2 \qquad \phi = x_1 \tag{1a}$$

$$\dot{x}_2 = u = f(\phi, \dot{\phi}) \qquad \dot{\phi} = x_2 \tag{1b}$$

where it is desired to determine the control input as a function of the state information ϕ and $\dot{\phi}$. Comparing the general shape of the switching characteristics in Fig. 1 with standard textbook optimal swirtching curves,[5,7,8] the weighted time-fuel and final-state control policy seemed to exhibit the best fit to the RLC switching strategy. The decision was then made

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*Engineer.

†Assistant Professor, Electrical Engineering Department. Member AIAA.

‡Associate Professor, Electrical Engineering Department.

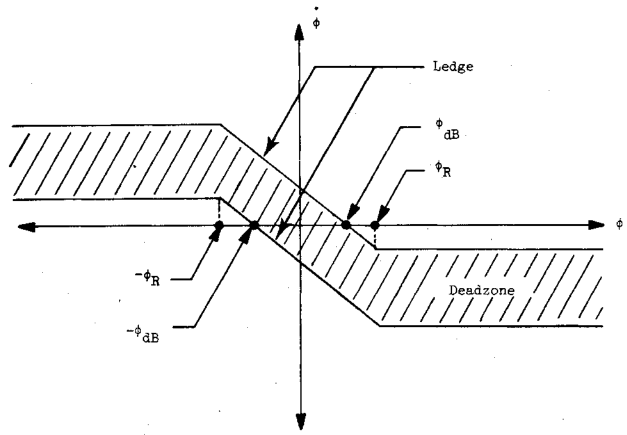

Fig. 1 Phase plane portrait of a single-axis switching curve of the type used with a six-engine reaction control jet attitude control law.

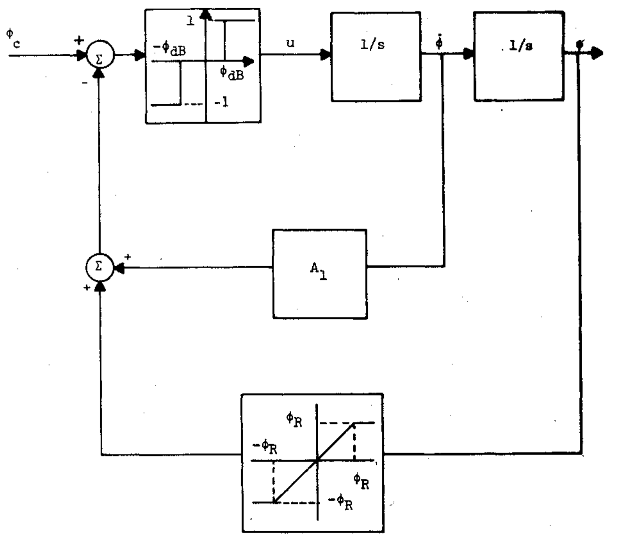

Fig. 2 Single-axis reaction control jet control system implementing the switching curve shown in Fig. 1.

to develop a relationship so as to generalize the empirical curve of Fig. 1 to an optimal weighted time-fuel law. The performance measure used was

$$J=\left\{\int_{t_0}^{t_f}(\lambda+|u|)\,dt\right\} \tag{2}$$

with the constraint

$$|\phi(t_f)|\leq\phi_{dB}$$

where ϕ_{dB} is a relative indicator parameter of the RLC deadband limits. Since the final position is constrained to lie within the deadzone limits, this problem can be solved for $\phi(t_f)=0$ and the results shifted to the limits $\phi(t_f)=\phi_{dB}$ and $\phi(t_f)=-\phi_{dB}$ for the solution to the deadzone problem.

The Hamiltonian formulation for this control problem is

$$H=\lambda+|u|+p_1\dot{\phi}+p_2u \tag{3}$$

p_1, p_2 being costates corresponding to ϕ, $\dot{\phi}$, and λ a time weight factor. Utilizing Pontryagin's Minimum (Maximum) Principle, the form of the optimal control can be shown to be[7]

$$\hat{\phi}(t)=-\tfrac{1}{2}\hat{\dot{\phi}}(t)^2+C_1 \qquad u=-1$$

$$\hat{\phi}(t)=\tfrac{1}{2}\hat{\dot{\phi}}(t)^2+C_2 \qquad u=1$$

$$\hat{\phi}(t)=C_3t+C_4 \qquad u=0$$

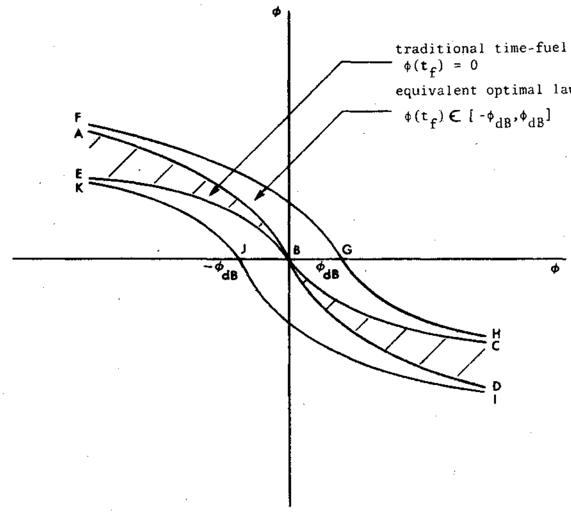

Fig. 3 Comparison of the traditional time-fuel optimal control law with $\phi[t_f]=0$ with the optimal law considered in the paper.

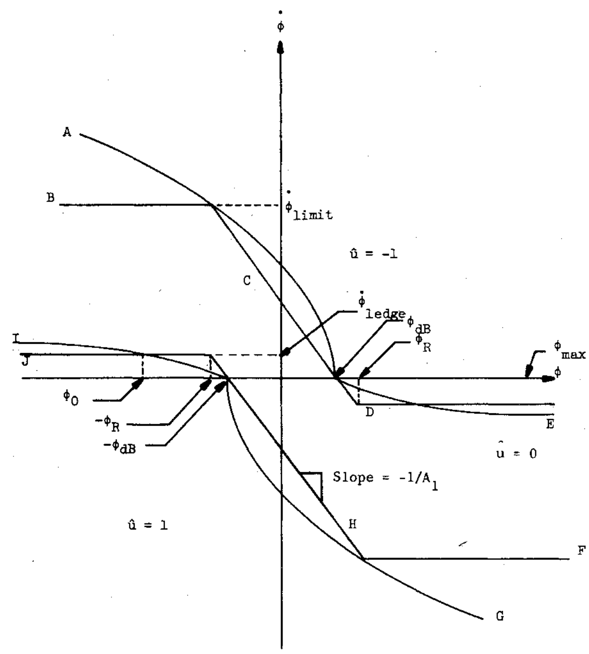

Fig. 4 Phase plane of the RLC control law and the corresponding optimal control law.

C_1, C_2, C_3, C_4 are constants. The optimal phase-plane switching curve for $\hat{u}$ going from $\hat{u}=+1$ to $\hat{u}=0$ is

$$\hat{\phi}(t)=-(\lambda+4)/(2\lambda)\;\hat{\dot{\phi}}(t)^2 \tag{4}$$

and the switching curve for u going from $\hat{u}=-1$ to $u=0$ is

$$\hat{\phi}(t)=[(\lambda+4)/(2\lambda)]\hat{\dot{\phi}}(t)^2 \tag{5}$$

These results are analogous to the optimal switching curves for the minimum time problem, which are

$$\hat{\phi}(t)=-\tfrac{1}{2}\,\hat{\dot{\phi}}(t)^2 \qquad u:\ +1,\ -1 \text{ switch} \tag{6}$$

$$\hat{\phi}(t)=\tfrac{1}{2}\,\hat{\dot{\phi}}(t)^2 \qquad U:\ -1,\ +1 \text{ switch} \tag{7}$$

Note that as $\lambda\to\infty$, Eqs. (4) and (5) approach Eqs. (6) and (7) respectively. These results confirm the intuitive feeling that as $\lambda\to\infty$, the time-fuel law approaches the minimum time problem when $\phi(t_f)=0$ is a constraint.

To analyze the original RLC problem, these optimal results are shifted to $\phi(t_f) = \pm\phi_{dB}$ by an appropriate change in the variable ϕ. This shift is introduced to match the terminal constraint $\phi(t_f)\ \epsilon[-\phi_{dB}, \phi_{dB}]$ so as to be analogous to the RLC controller. As shown in Fig. 3, this results in a change in the optimal switching boundaries from that of A-B-C-D-B-E for the traditional weighted time-fuel problem with $\phi(t_f)=0$ to F-G-H-I-J-K for the case being considered.

Figure 4 reveals that the sloping deadzone curves reach their limiting rate values for $\phi = \pm\phi_R$. In the $\hat{u}=0$ region for $|\phi| > \phi_R$, $\dot{\phi}_{ledge} \leq |\dot{\phi}| \leq \dot{\phi}_{lim}$. A commanded vehicle attitude ϕ_c is input and $2\phi_{dB}$ is the width of the deadzone about ϕ_c. For simplicity, ϕ_c is shown as zero in Fig. 4. The curves would be offset to the left or right by an amount ϕ_c in general, however. The constant A_1 represents the rate feedback control parameter shown in Fig. 2. For notational simplicity it is assumed $\dot{\phi}_{ledge}, \dot{\phi}_{limit}, \phi_R, \phi_{dB}, \phi_0$ are all greater than zero.

III. Derivation of an Optimal Weighted Time-Fuel Equivalent of an RLC Control Law

The rate ledge and rate limit can now be related to the states (which are related to λ) and the resulting expressions solved for that value of λ which yields a weighted time-fuel curve intersecting the rate ledge at $\phi=\phi_0$, as shown in Fig. 4. For $|\phi| \leq \phi_R$ the equations for the rate ledge switching curves using Figs. 2 and 4 are

Curve	*Defining Equation*	
C	$\phi + A_1\dot{\phi} = \phi_{dB}$	(8)
H	$\phi + A_1\dot{\phi} = -\phi_{dB}$	(9)

If $|\phi| > \phi_R$ the deadzone boundary equations become

D	$\dot{\phi} = -\dot{\phi}_{ledge}$	(10)
F	$\dot{\phi} = -\dot{\phi}_{limit}$	(11)
B	$\dot{\phi} = \dot{\phi}_{limit}$	(12)
J	$\dot{\phi} = \dot{\phi}_{ledge}$	(13)

The corresponding optimal switching curves are

A	$\hat{\phi} = -\frac{1}{2}\hat{\dot{\phi}}^2 + \phi_{dB}$	(14)
E	$\hat{\phi} = [(\lambda+4)/(2\lambda)] + \phi_{dB}$	(15)
G	$\hat{\phi} = \frac{1}{2}\hat{\dot{\phi}}^2 - \phi_{dB}$	(16)
I	$\hat{\phi} = -[(\lambda+4)/(2\lambda)] - \phi_{dB}$	(17)

Substituting the appropriate limiting value of ϕ (either ϕ_R or $-\phi_R$) into Eqs. (8), (9), (14), and (16) and solving these equations for $\dot{\phi}$ yields a relation between the controller parameter A_1 and the given RLC characteristics

$$A_1 = \left[\frac{\phi_r + \phi_{dB}}{2}\right]^{1/2} \tag{18}$$

The rate limit and rate ledge are obtained from (8) by solving for $\dot{\phi}$ with $\phi = \pm\phi_R$,

$$\dot{\phi}_{limit} = [2(\phi_R + \phi_{dB})]^{1/2} \tag{19}$$

$$\dot{\phi}_{ledge} = (\phi_R - \phi_{dB})\ [2/(\phi_R + \phi_{dB})]^{1/2} \tag{20}$$

The optimal switching curve corresponding to Eq. (20) is given by Eq. (4). Solving for $\dot{\phi}^2$, and including the shift from the origin to $-\phi_{dB}$, yields

$$\dot{\phi}^2 = -\frac{2\lambda}{\lambda+4}(\phi + \phi_{dB}) \tag{21}$$

Squaring Eq. (20) and equating to Eq. (21) for the case $\dot{\phi} = \dot{\phi}_{ledge}$ yields an expression corresponding to the intersection of the rate ledge of the RLC and the optimal switching curve which may then be solved for λ or ϕ_0, where ϕ_0 is the value ϕ where the rate ledge and optimal curves meet, as shown in Fig. 4

$$\phi_0 = -\left[\frac{\lambda+4}{\lambda}\right]\frac{(\phi_R - \phi_{dB})^2}{\phi_R + \phi_{dB}} - \phi_{dB} \tag{22}$$

$$\lambda = \frac{-4(\phi_R - \phi_{dB})^2}{(\phi_R - \phi_{dB})^2 + (\phi_0 + \phi_{dB})(\phi_R + \phi_{dB})} \tag{23}$$

The optimal weighted time-fuel control law that most closely corresponds to a particular RLC law may be determined by minimizing the mean-square rate errors between the two sitching curves. The value of λ determined minimizes the mean-square error for a given value of ϕ_{max}, the largest expected attitude error. The square error $f(\lambda,\phi)$ is defined to be

$$f(\lambda, \phi) = (\dot{\phi}_{optimal} - \dot{\phi}_{rate\ ledge})^2 \tag{24}$$

and $f(\lambda,\phi)$ is to be evaluated along curve I of Fig. 4. Evaluation along arc G in Fig. 4 is not performed since it is fixed by its general form and the fact that it passes through $(\phi,\dot{\phi}) = (-\phi_{dB}, 0)$ and $(\phi_R, -\dot{\phi}_{limit})$. Solving Eq. (21) gives the expression for $\dot{\phi}_{optimal}$ as

$$\dot{\phi}_{optimal} = \left[-\frac{2\lambda}{\lambda+4}(\phi + \phi_{dB})\right]^{1/2} \tag{25}$$

$\dot{\phi}_{ledge}$ is obtained from Eqs. (8), (9), and (20) as

$$\dot{\phi}_{ledge} = \begin{cases} -(\phi + \phi_{dB})\left[\dfrac{2}{\phi_R + \phi_{dB}}\right]^{1/2} & -\phi_R < \phi < -\phi_{dB} \\ (\phi_R - \phi_{dB})\left[\dfrac{2}{\phi_R + \phi_{dB}}\right]^{1/2} & -\phi_{max} < \phi < -\phi_R \end{cases} \tag{26}$$

The value of λ minimizing Eq. (24) is obtained from the solution of

$$\frac{\partial}{\partial\lambda}\left[\frac{1}{\phi_{max} - \phi_{dB}}\int_{-\phi_{dB}}^{-\phi_{max}} f(\lambda,\phi)\,d\phi\right] = 0 \tag{27}$$

and represents the "best-fit" equivalent optimal curve for Fig. 1. The integral in Eq. (27) must be divided into parts since $\dot{\phi}_{ledge}$ is defined by two different expressions over $[-\phi_{dB},$

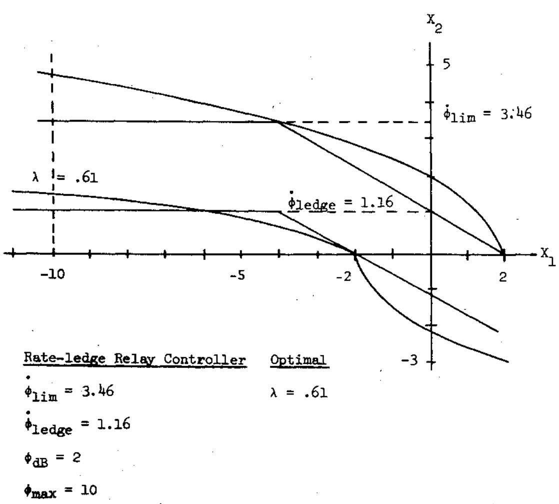

Fig. 5 Phase plane switching curves for example systems.

$-\phi_{max}]$. Substituting Eqs. (25) and (26) into Eq. (24) yields

$$f(\lambda,\phi)=\begin{cases} f_1(\lambda,\phi) & -\phi_R<\phi<-\phi_{dB} \\ f_2(\lambda,\phi) & -\phi_{max}<\phi<-\phi_R \end{cases} \tag{28}$$

where

$$f_1(\lambda,\phi)=\frac{2\lambda(\phi+\phi_{dB})}{(\lambda+4)} +4(\phi+\phi_{dB})\left[\frac{-\lambda(\phi+\phi_{dB})}{(\lambda+4)(\phi_R+\phi_{dB})}\right]^{1/2} +2\frac{(\phi-\phi_{dB})^2}{\phi_R+\phi_{dB}}$$

$$f_2(\lambda,\phi)=\frac{-2\lambda(\phi+\phi_{dB})}{\lambda+4} -4(\phi_R-\phi_{dB})\left[\frac{-\lambda(\phi+\phi_{dB})}{(\lambda+4)(\phi_R+\phi_{dB})}\right]^{1/2} +2\frac{(\phi_R-\phi_{dB})^2}{\phi_R+\phi_{dB}}$$

Using Eq. (28), Eq. (27) then becomes

$$\frac{\partial}{\partial\lambda}\left[\int_{-\phi_R}^{-\phi_{dB}} f_1(\lambda,\phi)\,d\phi+\int_{-\phi_{max}}^{-\phi_R} f_2(\lambda,\phi)\,d\phi\right]=0 \tag{29}$$

Evaluation of Eq. (29) using Leibniz rule results in

$$\int_{-\phi_R}^{-\phi_{dB}}\frac{\partial f_1(\lambda,\phi)}{\partial\lambda}\,d\phi+\int_{-\phi_{max}}^{-\phi_R}\frac{\partial f_2(\lambda,\phi)}{\partial\lambda}\,d\phi=0 \tag{30}$$

By obtaining the partial derivatives with respect to λ of f_1 and f_2 in Eq. (28), performing the indicated integration in Eq. (30), and performing a series of algebraic reductions, it can be shown[9] that the λ minimizing Eq. (24) is

$$\lambda=\frac{4K_1^2}{K_2^2-K_1^2} \tag{31}$$

$$K_1=\frac{\frac{16}{3}(\phi_R-\phi_{dB})(\phi_{max}-\phi_{dB})^{3/2}-\frac{32}{15}(\phi_R-\phi_{dB})^{5/2}}{(\phi_R+\phi_{dB})^{1/2}} \tag{32}$$

$$K_2=4(\phi_{max}-\phi_{dB})^2 \tag{33}$$

The significance of Eq. (31) is that the resulting λ, characterizing the optimal weighted time-fuel control law, is only a function of the a priori known RLC parameters. Hence, given an RLC control law, one can obtain directly the optimal weighted time-fuel equivalent in the sense of Eq. (24). This transform is guaranteed to exist because $\phi_{max}>\phi_R>\phi_{dB}$ so K_1, K_2 exist for all admissible ϕ_{max}, ϕ_R, ϕ_{dB}, hence λ in Eq. (31) exists. It is unique because Eqs. (32) and (33) yield only one solution of each admissible set ϕ_{max}, ϕ_R, ϕ_{dB}.

IV. Derivation of an RLC Equivalent of an Optimal Weighted Time-Fuel Controller

Now consider the reverse problem, that of determining the equivalent RLC law given an optimal weighted time-fuel law. The reason for studying this problem is that analytically a given optimal time-fuel law may be determined as the best for a given application, but system requirements may exclude its implementation but allow use of an RLC system composed of simple control elements.

Analogous to Eq. (29), the expression to be minimized now becomes

$$\frac{\partial}{\partial\phi_R}\left[\frac{1}{\phi_{max}-\phi_{dB}}\int_{-\phi_{dB}}^{\phi_{dB}} f(\phi_R,\phi)\,d\phi\right]=0 \tag{34}$$

and the quantities are as defined in Fig. 5. Application of the same procedure for determining λ results in a mathematically intractable sixth-degree expression in ϕ_R. A simple numerical approach is offered as an alternative. Using Eq. (22)

$$\phi_R^2+a_1\phi_R+a_0=0 \tag{35}$$

where

$$a_1=-2\phi_{dB}+\frac{\lambda}{\lambda+4}(\phi_0+\phi_{dB}) \tag{36}$$

$$a_0=\phi_{dB}^2+\frac{\lambda}{\lambda+4}(\phi_0+\phi_{dB})\phi_{dB} \tag{37}$$

the roots of which are

$$\phi_R=[-a_1\pm(a_1^2-4a_0)^{1/2}]/2 \tag{38}$$

As an initial guess of ϕ_0 in Eqs. (36-38), a reasonable starting value is

$$\phi_0=(-\phi_{dB}-\phi_{max})/2 \tag{39}$$

Using Eq. (33), the desired value of K_1, referred to as K_{1_d}, is computed from

$$K_{1_d}=K_2[\lambda/(\lambda+4)]^{1/2} \tag{40}$$

Performing a Taylor Series expansion of K_1 in Eq. (32) about the point ϕ_R and truncating after the linear terms yields

$$K_1(\phi_R)=K_1(\phi_{R_0})+K_1'(\phi_{R_0})\delta\phi_R \tag{41}$$

where

$$\delta\phi_R=\phi_R-\phi_{R_0} \tag{42}$$

$$B=\frac{(\phi_{R_0}+\phi_{dB})^{1/2}-0.5(\phi_R+\phi_{dB})^{-1/2}(\phi_{R_0}-\phi_{dB})}{(\phi_{R_0}+\phi_{dB})} \tag{43}$$

$$C=\frac{2.5(\phi_{R_0}-\phi_{dB})^{3/2}(\phi_{R_0}+\phi_{dB})^{1/2}}{(\phi_{R_0}+\phi_{dB})} \tag{44}$$

$$D=0.5(\phi_{R_0}+\phi_{dB})^{-1/2}(\phi_{R_0}-\phi_{dB})^{5/2}/(\phi_{R_0}+\phi_{dB}) \tag{45}$$

$$K_1'(\phi_{R_0})=\frac{d}{d\phi_R}$$

$$K_1=\frac{16B}{3}(\phi_{max}-\phi_{dB})^{3/2}-\frac{32}{15}(C-D) \tag{46}$$

Solving Eq. (41) for $\delta\phi_R$

$$\delta\phi_{R_0}=\frac{K_1(\phi_R)-K_1(\phi_{R_0})}{K_1'(\phi_{R_0})}=\frac{K_{1_d}-K_1(\phi_{R_0})}{K_1'(\phi_{R_0})} \tag{47}$$

where K_1 is computed using Eq. (32), K_1' computed using Eq. (46).

Using the previous results a recursive algorithm for determining ϕ_R is suggested.

Initialization

$\phi_0(0)$ from Eq. (39);

$\phi_R(0)$ from Eq. (38);

K_2 from Eq. (33);

K_{1_d} from Eq. (40).

Recursive Algorithm

$$K_1(k+1)=\left[\frac{16}{3}(\phi_R(k)-\phi_{dB})\phi_{max}-\phi_{dB})^{3/2}-\frac{32}{15}(\phi_R(k)-\phi_{dB})^{5/2}\right]\div(\phi_R(k)+\phi_{dB})^{1/2} \quad (48)$$

$$B(k)=[(\phi_R(k)+\phi_{dB})^{1/2}-.5(\phi_R(k)+\phi_{dB})^{-1/2}(\phi_R(k)-\phi_{dB})]\div(\phi_R(k)+\phi_{dB}) \quad (49)$$

$$C(k)=2.5(\phi_R(k)-\phi_{dB})^{3/2}(\phi_R(k)+\phi_{dB})^{1/2}\div(\phi_R(k)+\phi_{dB}) \quad (50)$$

$$D(k)=.5(\phi_R(k)+\phi_{dB})^{-1/2}(\phi_R(k)-\phi_{dB})^{5/2}\div(\phi_R(k)+\phi_{dB}) \quad (51)$$

$$K_1'(k)=\frac{16}{3}(\phi_{max}-\phi_{dB})^{3/2}B(k)-\frac{32}{15}(C(k)-D(k) \quad (52)$$

$$\delta\phi_R(k)=\frac{K_{1_d}-K_1(k)}{K_1'(k)} \quad (53)$$

$$\phi_R(k+1)=\phi_R(k)+\delta\phi_R(k) \quad (54)$$

and $\phi_R(j)$, etc. represents the jth iterative computation, $j=0,1,2,\ldots$. Experience has demonstrated rapid (≤ 2 iterations in practice) convergence of the above algorithm.

To demonstrate the existence and uniqueness of the optimal to RLC transformation, it must merely be shown that the resulting ϕ_R from Eq. (38) exists and is unique for all admissible RLC control laws. This is because the given optimal law already has defined λ, ϕ_{dB} inasmuch as arcs A-E pass through $(\phi, \dot\phi)=(\phi_{dB},0)$ and arcs I-G pass through $(\phi,\dot\phi)=(-\phi_{dB},0)$. Also, $\dot\phi_{limit}$ is uniquely defined if ϕ_R is uniquely defined. Therefore, these parameters ϕ_{dB}, $\dot\phi_{limit}$, λ, along with ϕ_R are sufficient to uniquely define the RLC switching curves B-C-D and J-H-F in Fig. 4.

It has been shown that $\lambda>0$, $\phi_R\geq\phi_{dB}$, and $\phi_0\leq-\phi_{dB}\leq 0$. Defining $\alpha=[\lambda/(\lambda+4)]$ it follows that $0<\alpha\leq 1$. Using $\phi_0=-(\phi_{dB}+\epsilon)$, $\epsilon\geq 0$ and Eqs. (36) and (37)

$$a_0=\phi_{dB}^2-\alpha_{dB};\quad a_1=-2\phi_{dB}-\alpha\epsilon$$

From Eq. (38)

$$(a_1^2-4a_0)^{1/2}=\alpha\epsilon+\beta;\qquad \beta>0$$

resulting in

$$\phi_{R_1}=(2\phi_{dB}+\alpha\epsilon)\ +\ (\alpha\epsilon+\beta)/2$$

$$\phi_{R_2}=(2\phi_{dB}+\alpha\epsilon)-(\alpha\epsilon+\beta)/2];\qquad <\phi_{dB}$$

but $\phi_R>\phi_{dB}$ and ϕ_{R_2} contradicts this. Therefore, ϕ_R exists and is unique. Q.E.D.

Table 1 Comparison of the optimal and RLC equivalent control laws from example with respect to fuel usage (20 sec simulation time; sufficient to drive systems into deadzone)

Initial Condition $(\phi,\dot\phi)$	Optimal weighted time-fuel law	Rate-ledge relay	% Difference
(1,1)	1.035	0.99349	4.22
(1,−1)	1.05	0.999	5.10
(1,−3)	4.2	4.386	4.24
(−6,3.5)	3.546	3.505	1.17
(−6,2)	2.041	2.00	2.05
(−6,−1)	3.08	3.316	0.25
(3,−2)	2.04	2.0	2.00
(3,1.7)	3.13	3.84	18.48
(3,3)	5.303	5.311	0.15

V. Example and Simulation Results

Illustrations of implementation of the two transformations will now be given as well as simulation results to reinforce the idea of comparable response results. Given an RLC law with $\phi_{dB}=2$, $\phi_R=4$, $\phi_{max.}=10$, determine the equivalent optimal law using Eq. (2) for the criterion in Eq. (24). Using Eqs. (31-33), $K_1=93$, $K_2=256$, $\lambda=0.61$.

Now consider the reverse procedure, given the previous values of λ, ϕ_{dB}, and ϕ_{max}. Using Eq. (39), $\phi_0=-6$, which, coupled with Eq. (38) yields $\phi_{R_0}=3.74$. From Eq. (33), $K_2=256$ and from Eq. (40), $K_{1_d}=93$. Using Eq. (32), $K_1(0)=83.5$ and from Eq. (46) $K_1'(0)=37.8$ yielding $\delta\phi_R=0.252$ from Eq. (47). Updating ϕ_R using Eq. (54) results in $\phi_R(1)=3.992$ which is within 0.2% of the actual ϕ_R. This value of ϕ_R is then used to determine A_1, $\dot\phi_{lim}$, $\dot\phi_{ledge}$ from Eqs. (18-20) respectively. The computed values are $A_1=1.73$, $\dot\phi_{lim}=3.46$, $\dot\phi_{ledge}=1.16$. The corresponding rate ledge and optimal time-fuel switching curves for this example are shown in Fig. 5.

Simulations of the two control laws were implemented to compare response results using various initial conditions. Table 1 compares the fuel usage between the RLC and optimal weighted time-fuel laws. It is clear from the very small per cent differences in fuel usage that the two methods represent virtually the same system, which tends to verify the transformations developed.

VI. Conclusions

Under the assumption of second-order dynamics, a technique for obtaining an "equivalent" optimal weighted time-fuel control law given a rate-ledge relay control law of the type in Fig. 1 was developed. The technique was based on minimizing the mean-square error between the switching curves. Conversely, a reverse transformation, from an optimal law to an RLC equivalent was derived. Both resulting transformations were shown to insure the existence and uniqueness of a solution. The simplicity of the transformations, coupled with design utility (RLC to optimal) as well as east of implementation (optimal to RLC) make the methods attractive for use in the design and analysis of spacecraft attitude control systems.

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